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Math Club Teaser #34

November 6, 2009
(due Nobember 13, 2009)

Solution

The 2-, 4-, 6-, and 8-digit numbers are even, so the digits $\{2, 4, 6, 8\}$ appear in positions 2, 4, 6, 8 in some order. Also, the 5-digit number is divisible by five, so the 5^{th} digit must be 5. This leaves the following options (each box shows the digits that we can try in that position):

1 3 7 9	2 4 6 8	1 3 7 9	2 4 6 8	5	2 4 6 8	1 3 7 9	2 4 6 8	1 3 7 9

A number is divisible by four if its last two digits form a number divisible by four. In our situation, the choices are 12, 16, 32, 36, 72, 76, 92, 96. In other words, the 4^{th} and 8^{th} digits must be 2 and 6 in some order, leaving the digits 4 and 8 for positions 2 and 6:

1 3 7 9	4 8	1 3 7 9	2 6	5	4 8	1 3 7 9	2 6	1 3 7 9

A number is divisible by three if its digits add up to a multiple of three. In our situation, boxes $(1, 2, 3)$, boxes $(4, 5, 6)$, and boxes $(7, 8, 9)$, each must add to a multiple of three. Because of this, boxes $(1, 2, 3)$ hold 147, 183, 189, 381, 387, 741, 783, 789, 981, or 987 ; boxes $(4, 5, 6)$ hold 258 or 654; and boxes $(7, 8, 9)$ hold 123, 129, 321, 327, 369, 723, 729, 921, 927, or 963. These choices interfere with each other, so the only possible 9-digit numbers are

147258369	147258963	183654729	183654927	189654327
189654723	381654729	381654927	387654129	387654921
741258369	741258963	783654129	783654921	789654123
789654321	981654327	981654723	987654123	987654321

and only $\boxed{381654729}$ satisfies the divisibility by 7 and 8 requirements.

Solved by:

Captain Nemo, Toni Hayward, Phoenix, The Residues, Think Before you Act.