I U P U I
Math Club Teaser #45

September 3, 2010
(due September 10, 2010)

Solution

You can easily check that a number n with the desired property cannot have only one or two digits. Let's see what happens if we assume n has three digits:

If the digits are a, b, c, write n as $\overline{abc}$; this means $n = 100a + 10b + c$. The criterion for $\overline{abc}$ to be divisible by 11 is that b equals $a + c$ or $a+c-11$. In the first case, $\overline{abc}/11 = \overline{ac}$, so $a^2 + (a+c)^2 + c^2 = 10a + c$. Note that the sum of three squares must be even, so c is 0, 2, 4, 6, or 8. Checking these 5 possibilities gives only one solution: 550.

In the second case, we have $a^2 + (a + c - 11)^2 + c^2 = 10(a - 1) + c$. Now the sum of squares must be odd, so c is 1, 3, 5, 7, or 9. Checking these five cases yields one more solution: 803.

The same type of argument works for four digits, and there turns out to be no new solutions. On the other hand, a number with five or more digits is too large to satisfy the condition, so the only valid solutions are

$$550\ ,\ 803.$$

Solved by:

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